and, therefore,

$$\begin{aligned} \Pi(\varphi_\gamma, \mu_\gamma) &= \Pi(\varphi'_{\gamma'}, \mu'_{\gamma'}) - \Pi(\varphi'_{\gamma'}, \lrcorner_{\gamma'} - \mu'_{\gamma'}) \\ &= \Pi(\varphi'_{\gamma'}, \mu'_{\gamma'}) + \Pi(\varphi'_{\gamma'}, \lrcorner_{\gamma'} + \mu'_{\gamma'}), \qquad (16) \end{aligned}$$

or, writing for brevity $\mu^{\backprime}$ for the co-amplitude of μ',

$$\Pi(\varphi_\gamma, \mu_\gamma) = \Pi(\varphi'_{\gamma'}, \mu'_{\gamma'}) - \Pi(\varphi'_{\gamma'}, \mu^{\backprime}_{\gamma'}); \quad \mu'_{\gamma'} + \mu^{\backprime}_{\gamma'} = \lrcorner_{\gamma'}. \qquad (16')$$

This I call Somoff's theorem. Somoff, however, makes no use of it in this form. He employs a relation which is obtained from (16) by expressing $\Pi(\varphi'_{\gamma'}, \lrcorner_{\gamma'} - \mu'_{\gamma'})$ in terms of $\Pi(\varphi'_{\gamma'}, \mu'_{\gamma'})$ by (6). We have thus

$$\begin{aligned} \Pi(\varphi_\gamma, \mu_\gamma) &= 2\Pi(\varphi'_{\gamma'}, \mu'_{\gamma'}) - \gamma'^2 \varphi'_{\gamma'} \sin\mu' \frac{\cos\mu'}{\Delta\mu'} + \tfrac{1}{2} l \frac{\Delta(\mu'_{\gamma'} - \varphi'_{\gamma'})_{-\gamma'}}{\Delta(\mu'_{\gamma'} + \varphi'_{\gamma'})_{-\gamma'}} \\ &= 2\Pi(\varphi'_{\gamma'}, \mu'_{\gamma'}) - \gamma\varphi_\gamma \sin\mu + \tfrac{1}{2} l \frac{1 + \gamma \sin\mu \sin\varphi}{1 - \gamma \sin\mu \sin\varphi}. \qquad (17) \end{aligned}$$

[TO BE CONTINUED.]

A SIMPLE DISCUSSION OF LOGARITHMIC ERRORS.

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[CONTINUED FROM VOL. II, PAGE 43.]

PROBLEM XI.

To find the probable error of an interpolated logarithmic trigonometric function.

Tables of logarithmic trigonometric functions may be divided into three classes. The first class embraces those in which the tabular difference is to be multiplied by a decimal of one, two, three, or, occasionally, four places. In this class we find 5, 6, and 7-place tables in which the functions are given for each 10 seconds or for each second, those tables in which the functions are given for each minute, and are interpolated for decimal parts of a minute, the 4-place tables in which the functions are given for each 10 minutes, and tables in which the differences of the successive values of the argument are tenths or hundredths of a degree.

The second class includes those tables in which the functions are given for

each minute, and the proportional parts for 6, 7, 8, 9, 10, 20, 30, 40, and 50 seconds. Gauss's or Hoüel's 5-place tables are such.

In the third class are those 5 and 6-place tables in which the functions are given for each minute, and the variations of the functions for each second, these variations being carried to two decimal places. In Olney's or Davies's 6 place tables these variations, though printed directly opposite the functions, should be placed half a line lower, since they have been so computed that they show the rate at which the function is varying at each half minute. In Schlömilch's 5-place tables the variations have been found by dividing the differences between successive tabular functions by 60.

Class I. The discussion of this class has already virtually been given in Problem I. If x be taken as 0.01, 0.02, . . . 0.99 the average probable value of $xc' + (1 - x)c$ becomes 0.810×0.25, and no appreciable change will be made if we have x equal to 0.001, 0.002, . . . 0.999. But taking into account the fact that x may be 0.0, in which case the probable error is 0.25, we find that, when x varies from 0.1 to 0.9, the average probable error is 0.31. When x varies from 0.01 to 0.99 and from 0.001 to 0.999, the average probable error is 0.32.

Class II. In Hoüel's 5-place table, the functions in the tables of proportional parts are given to the nearest integer only. If the number of seconds has two figures and does not end in a cipher, two interpolations from the table of proportional parts must be taken, and the probable error of the sum of these is $\sqrt{2}\,.(0.25)$, or 0.35. If n denote the number of seconds and $x = n/60$, the error of the interpolated function would be (aside from the source of error just mentioned), $xc' + (1 - x)c$. Since the probable values of c' and c are each 0.25, we have the following formula for the probable error in this particular case:—

$$\text{Error} = \sqrt{[x^2 + (1 - x)^2 + 2]}\,.(0.25).$$

When but a single number is taken from the table of proportional parts the proper formula is

$$\text{Error} = \sqrt{[x^2 + (1 - x)^2 + 1]}\,.(0.25).$$

Substituting the values of x, which range from 1/60 to 59/60 in the preceding formulæ, we find that the average of the probable errors is 0.39, the minimum being 0.31 and the maximum 0.43.

But, if the functions in the tables of proportional parts had been given to tenths, as in Gauss's table, and the interpolated functions had been carried to 6 places, besides the quantity $xc' + (1 - x)c$ there would have been, arising from the neglect of the 7th place, a probable error of 0.025 (in units of the 5th place) when the proportional part was given directly in the table, and of 0.035 if two entrances into the interpolation-table were made. This has so slight an effect that we pay no further attention to it, and assume that the error arising from

dropping the 6th place is 0.25. The last formula given above applies to this case, and for the average probable error we find 0.32.

Class III. In Olney's or Davies's 6-place tables the difference for one second has a probable error of 0.0025 in units of the 6th place. This is multiplied by numbers ranging from 0.1 to 59.9. The average probable error of the product is thus 30×0.0025, or 0.075. Combining this with the the probable error of the mantissa to which the product is added, and remembering that 0.25 is rejected when the result is limited to 6 places, we have

$$\text{Error} = \sqrt{[(0.25)^2 + (0.075)^2 + (0.25)^2]} = 0.36.$$

The errors arising in the use of Schlömilch's 5-place table are nearly the same in magnitude as those of the tables in Class I. Let D be the apparent tabular difference and n the number of seconds. We really multiply $D/60$ (carried to hundredths) by n. But this is practically equivalent to multiplying $n/60$ (carried to hundredths) by D. Proceeding as under Case I, and including the error 0.075 which was lately mentioned, we find for the result

$$\text{Error} = \sqrt{[(0.25)^2 + (0.810)^2 (0.25)^2 + (0.075)^2]} = 0.33.$$

Hence functions interpolated from Schlömilch's table have a less probable error than those taken from Olney's or Davies's. This accords with the conclusion reached in Problem VI.

PROBLEM XII.

Given the logarithm of the tangent of an angle, to find the probable error of the corresponding interpolated angle.

We divide the tables into the three classes given in Problem XI.

Class I. The interpolation in this class of tables is like that in tables of logarithms of natural numbers. Hence the solution of this problem is similar to that of Problem II. If but one figure is interpolated we have

$$\text{Error} = \sqrt{\left[\left(\frac{1.98}{D}\right)^2 + (0.25)^2\right]}.$$

If more than one figure is interpolated we have (see under Class I of Problem XI) instead of 1.98, 8.10 / 4, or 2.02. For this case

$$\text{Error} = \sqrt{\left[\left(\frac{0.202}{D} \cdot 10^f\right)^2 + (0.25)^2\right]}.$$

By substituting the values of D for different portions of the table, one can compute a series of values for the error.

Class II. Let us assume that the proportional parts are given to tenths, as in Gauss's table. Suppose that the angles are carried accurately to thousandths

of a minute by computing d/D. Since the neglect of the ten-thousandths would introduce a probable error of only $60'' \times 0.00025$, or $0''.015$ into the angle finally derived, and since the other errors are very much larger than this, we pay no further attention to it, but account the thousandths accurate. According to the reasoning in Problem II (remembering to use $0.810/4$ instead of $0.793/4$), we then have $0.202/D$ as the probable error. But, in reducing the angle to seconds, this error would be multiplied by $60''$, giving the expression $12''.12/D$. The error thus far treated of arises from the inaccuracy of the tabular logarithms. But in using the tables of proportional parts given in Gauss, further error is introduced, because the proportional parts are carried to tenths only. Now, in general, if an angle be carried to hundredths of a second, three subtractions will be made, and at each subtraction a probable error of 0.025 of a unit is introduced into the remainder. Hence the probable error of the final remainder is $0.025\sqrt{3}$. Since the change of the proportional part for $1''$ is $D/60$, the effect in seconds of the error just mentioned is $0.025\sqrt{3} \div D/60$, which equals $1.5\sqrt{3}/D = 2.6/D$. The final value of the angle being restricted to seconds, a further error, whose probable value is $0''.25$ is introduced. The three errors which we have discussed being independent of each other, the final result is

$$\text{Error} = \sqrt{\left[\left(\frac{12''.12}{D}\right)^2 + \left(\frac{2''.6}{D}\right)^2 + (0.25)^2\right]}.$$

Class III. In using Olney's or Davies's 6-place tables, the difference between the given logarithm and the tabular one has a probable error of 0.25. Denoting this difference by d, and the difference for $1''$ by D', since the latter has a probable error of 0.0025, we divide $d \pm 0.25$ by $D' \pm 0.0025$ and obtain the quotient

$$\frac{d}{D'} + \frac{0.25}{D'} \mp (0.0025)\frac{d}{D'^2} - \dots .$$

We really adopt the value d/D'. Hence the error is $0.25/D' \mp 0.0025\, d/D'^2$. If the result is restricted to tenths of a second, as usual, another error whose probable value is 0.025 is introduced. The average value of d/D' is 30, and the formula for the average probable error becomes

$$\text{Error} = \sqrt{\left[\left(\frac{0.25}{D'}\right)^2 + \left(\frac{0.075}{D'}\right)^2 + (0.025)^2\right]},$$

or if D is taken as the difference for one minute,

$$\text{Error} = \sqrt{\left[\left(\frac{15}{D}\right)^2 + \left(\frac{4.5}{D}\right)^2 + (0.025)^2\right]}.$$

In employing Schlömilch's table, if the difference for $1''$ were carried to 4 or 5 places, and the interpolated seconds to hundredths, the result would be the same

as if d were divided by D, the quotient carried to 4 or 5 places of decimals, and then multiplied by $60''$.

But in this case the error due to c and c' would be, according to Problem II,

$$\frac{d}{D^2}c' + \frac{1-\frac{d}{D}}{D}c,$$

whose probable value has been shown to be $0.202/D$ when expressed as a decimal of a minute, or $12.12/D$, when expressed in seconds. As in Olney's table there is another error of $0.075/D'$ due to the inaccuracy of D'; when the interpolated seconds are not carried beyond units, 0.25 is the probable value of the rejected figures. Thus the final formula is ($1''$ being the unit)

$$\text{Error} = \sqrt{\left[\left(\frac{0.202}{D'}\right)^2 + \left(\frac{0.075}{D'}\right)^2 + (0.25)^2\right]}$$
$$= \sqrt{\left[\left(\frac{12.12}{D}\right)^2 + \left(\frac{4.5}{D}\right)^2 + (0.25)^2\right]}.$$

PROBLEM XIII.

To find the maximum errors in Problem XI.

Class I. The maximum error is 1.0 in the n^{th} place for n-place logarithms, as shown in Problem IV.

Class II. For Hoüel's table no figure is rejected from the result given by interpolation. In addition to the error caused by the incorrectness of the tabular logarithms, one is introduced depending on the fact that each of the two functions taken from the table of proportional parts may have an error of 0.5 in the fifth place, and these errors may conspire. The formula then is

$$\text{Maximum error} = xc' + (1-x)c + 1.00$$
$$= 1.50.$$

If Gauss's table be used, the error due to cutting off the functions in the tables of proportional parts at tenths of a unit of the fifth decimal place, may amount to $0.05 + 0.05$, or 0.10. Since all figures beyond the fifth place are rejected, another error of 0.5 may arise. Hence we have

$$\text{Maximum error} = xc' + (1-x)c + 0.6$$
$$= 1.10.$$

Class III. In Olney's table an error arises from the neglect of second differences, which is often quite appreciable, but will not be treated of. The difference for $1''$ may have an error of 0.005 in units of the sixth place. The product of this by 59.9 (the largest number of seconds ever employed) is 0.30. The maximum error of the tabular mantissa to which the proportional part is applied is

0.50. The rejected figures beyond the sixth place may amount to 0.50. Hence

$$\text{Maximum error} = 1.30.$$

If Schlömilch's table be used, the error in the difference for $1''$, caused by terminating it at the second decimal place, may be $0.00\frac{1}{3}$ of a unit of the fifth place. This multiplied by 59 gives 0.20. This table is also subject to the error of those of Class I. Therefore

$$\text{Maximum error} = 1.20$$

PROBLEM XIV.

To find the maximum errors in Problem XII.

Class I. From Problem V we get

$$\text{Maximum error} = \frac{0.50}{\delta} \cdot 10^f + 0.50,$$

in which δ is the minimum value of D, and f denotes the number of figures interpolated. The error is given in units of the last interpolated figure. For angles taken from the logarithmic tangents of Bremiker's 6-place table the formula gives 1.7 tenths of a second, or $0''.17$.

Class II. We shall not discuss Hoüel's table, because the value of the interpolated seconds frequently depends upon the computer's judgment.

In using Gauss's table let us suppose that the angles are carried to hundredths of a second, the process being to divide the difference d by the tabular difference D, and multiply the result by 60. Then by Problem II the error in seconds is

$$60''\left[\frac{d}{D^2}c' + \frac{1-\frac{d}{D}}{D}c - \frac{1-\frac{d}{D}}{D^2}c\,(c'-c) + \frac{d}{D^2}c'\,(c'-c) + \ldots\right].$$

Since the first two terms of the parenthesis are much larger than the others, we take them to find the condition for a maximum. Plainly, c' must equal c. Substituting 0.50 for them, the foregoing expression becomes

$$60''\left(\frac{0.50}{D}\right),$$

which is a maximum when D is a minimum. The least value of D in the tangent column is 25, and then the above expression $= 1''.20$. But the employment of Gauss's tables of proportional parts causes another error, because the functions in them are carried to tenths only. A little experimentation shows that the maximum error due to this cause, when D is near 25, is $0''.20$. In restricting the angle to the nearest second a possible error of $0''.50$ is made, which added to those previously given makes $1''.90$. It is hardly possible that this limit of inaccuracy is ever reached in practice.

Class III. In Olney's table we assume that the difference for $1''$ has *only* the error 0.005, and that the next less tabular mantissa is 0.5 in error in the 6th place. Then Problem XII gives us

$$\text{Maximum error} = \frac{0''.50}{\delta'} + \left(0''.005\,\frac{d}{\delta'}\right) \div \delta' + 0''.05,$$

in which δ' is the minimum value of D', and the interpolated seconds are terminated at tenths. Since $\delta' = 4.21$, the error is $0''.24$.

In employing Schlömilch's table there are three errors, the first due to the inaccuracy of the tabular logarithms, the second to the cutting off of D' at hundredths, and the third caused by restricting the interpolated angle to the nearest second. From previous discussions we readily get

$$\text{Maximum error} = 1''.20 + \left(0''.003\tfrac{1}{3}\,\frac{d}{\delta'}\right) \div \delta' + 0''.50.$$

Assuming that $d/\delta' = 57$, and $\delta' = 0.42$, this reduces to $2''.15$. In practice this error is never reached, and it is doubtful whether the limit $2''.00$ can be attained.

PROBLEM XV.

To compare the theories of Problems XI and XII with the facts.

In testing the theories of these Problems, Bremiker's 6-place table was used as a specimen of those of Class I. 120 angles uniformly distributed between 5° and 65°, and carried to seconds, were chosen, and their logarithmic tangents taken from each of the five tables given below. The results were compared with those obtained from Schrön's 7-place table, which was used as a standard. The averages of the errors, in units of the n^{th} place, for n-place logarithms are given below: —

	Actual Errors.	Theoretical Errors.
Bremiker,	0.27	0.31
Hoüel,	0.39	0.39
Gauss,	0.31	0.32
Olney,	0.36	0.36
Schlömilch,	0.31	0.33

The average of the errors of the functions taken from Bremiker was unusually small.

In testing the theories of Problem XII, 120 logarithmic tangents were taken, and the corresponding angles, which ranged from 35° to 45°, were obtained. Schrön was used again as the standard. For Bremiker the average value of D was 43, and for Gauss, 26. For Olney and for Schlömilch the aver-

age values of D' were 4.30 and 0.43 respectively. The average errors are tabulated below: —

	Actual Error.	Theoretical Error.
Bremiker,	0''.050	0''.053
Gauss,	0.45	0.54
Olney,	0.063	0.066
Schlömilch,	0.50	0.56

Being dissatisfied with the discrepancies in the cases of Gauss and Schlömilch, I worked out another set of 120 for each, taking care to have the resulting seconds quite uniformly distributed throughout the minute. Gauss's error became 0''.53, but Schlömilch's was 0''.49 for forward interpolation and 0''.52 for backward interpolation. The seconds given by the interpolation backward differed by a unit from those given by the forward, in 61 of the 120 angles; this portion of the table seems to be bewitched.

REMARK.—As it may be of interest to know the different times required in using various tables, I append those for obtaining the 120 angles from their logarithmic tangents. Schrön's angles were carried to hundredths of a second, Bremiker's and Olney's to tenths, and those from the remaining tables to seconds. The degrees and minutes were written down in Schrön's angles only, the seconds alone being recorded for the other tables. A multiplication table was used to facilitate interpolation from Olney. The seconds of each angle were found twice, once by forward interpolation, and once by backward, to insure their accuracy.

	First 40.		Second 40.		Third 40.		Total.		
	m.	*sec.*	*m.*	*sec.*	*m.*	*sec.*	*h.*	*m.*	*sec.*
Schrön (7-place),	27	40	25	0	21	10	1	13	50
Bremiker (6-place),	14	0	12	15	13	45	0	40	0
Olney (6-place),	25	0	21	40	16	35	1	3	15
Gauss (5-place),	10	45	10	20	9	35	0	30	40
Schlömilch (5-place),	12	40	11	30	11	10	0	35	20
Gernerth (5-place),	5	15	4	25	4	30	0	14	10

In Gernerth's table the functions are given for each 10''.

NOTE.—Since logarithmic errors do not follow with exactness the laws of those treated of in the method of least squares, investigations based on the law assumed at the beginning of this series of articles cannot be expected to give results rigorously true; but their close approach to the truth, together with the difficulties of a rigid investigation of the subject, seem to justify the approximate method here employed.